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COMMENT

Some comments on the application of analytic regularisation to the Casimir forces†

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Abstract. Expressing the generalised zeta function $\Sigma_n \omega_n^{-s}$ in terms of the 'partition function' $\Sigma_n e^{-\alpha\omega_n}$, we show in this paper why we obtain the same answer for the Casimir effect (in many physical situations) if we use, alternatively, the analytic continuation procedure for the zeta function or take the first derivative of the above partition function and make $\alpha \to 0^+$ with the neglect of all pole terms in $\alpha = 0$.

1. Introduction

In a previous paper (Ruggiero *et al* 1977) we have discussed, by using procedures similar to those presented by Gelfand and Shilov in their book (1962), the application of analytic regularisation to the Casimir effect (Casimir 1948, Fierz 1960, Boyer 1968). In particular, the exponential cut-offs used by these last authors were interpreted by us as analytic regulators.

More explicitly, by considering the zero-point energy of the system:

$$E_0 = \frac{\hbar}{2}\sum_n \omega_n = -\frac{\hbar}{2}\lim_{\alpha\to 0^+}\frac{\partial}{\partial\alpha}\sum_n e^{-\alpha\omega_n} \tag{1.1}$$

where n denotes all the relevant quantum numbers of the system, the prescription which we have used in order to obtain the finite part of the left-hand side of equation (1.1) was to subtract all the poles of the corresponding right-hand side. This was done by considering α as an analytic regulator for $\mathrm{Re}\,\alpha > 0$ and by imitating a similar situation which occurs in quantum electrodynamics where we perform analytic regularisation in the manner of Gelfand and Shilov (1962). (For more details see Bollini *et al* (1964).)

We have also considered in our previously quoted paper analytic regularisation by means of the use of generalised zeta functions:

$$\frac{\hbar}{2}\sum_n \omega_n^{-s}. \tag{1.2}$$

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The physical zero-point energy would be obtained by performing the analytic continuation of expression (1.2) up to $s=-1$. A similar suggestion has been made by Hawking (1977). As examples, we have discussed the Casimir effect in rectangular systems, in one, two and three dimensions (the generalisations to any dimension being trivial from our paper) and in a system of two conducting parallel plates. We have also studied the meaning of equation (1.2) from the point of view of analytic continuation of the Green functions for the system.

As an interesting by-product, it came out that the analytic regularisation by means of generalised zeta functions produced values for the zero-point energies which did not present poles at $s=-1$, giving automatically finite results, at least for the examples which we have treated. It was not necessary to do further subtractions, as happens in quantum field theory (Bollini *et al* 1964).

It was also seen that the finite parts, in the examples treated by us, came out to be the same by using the two methods described above, but we did not discuss in our previous paper why the results are the same. A similar problem appears in the study of the moments $\mathrm{Tr}(H^{-s})$, where H describes the Hamiltonian of a non-relativistic particle in the s-state in a spherical potential $V(r)$ (short range and analytic in r). Buslaev and Faddeev (1960) have studied the moments $\mathrm{Tr}(H^{-s})$ for $s \leqslant 0$ by means of an analytic continuation of the generalised zeta function. Percival (1962) has used the expansion of the partition function $Z(-\beta)=\mathrm{Tr}(\mathrm{e}^{-\beta H})$, and by taking successive derivatives of it with respect to β and letting $\beta \to 0^+$, he obtained the different energy moments for integer $s \leqslant 0$; as the moments come out divergent for $\beta \to 0^+$, Percival has defined a new derivative operator which amounts to taking the usual derivative (of any order) and subtracting from it all the singular terms for $\beta \to 0^+$. In this way he obtained the same results as Buslaev and Faddeev. By expressing the generalised zeta function in terms of the partition function it is simple to see why the two procedures give the same answer (Pimentel and Zimerman 1978).

The same method will be used in this paper.

We can express the generalised zeta function $\Sigma_n \omega_n^{-s}$ in terms of the 'partition function' $\Sigma_n \mathrm{e}^{-\omega_n x}$ by means of a Mellin transform, valid for $\mathrm{Re}\, s > a$ (a is some real number), and can do the analytic continuation for $\mathrm{Re}\, s < a$ (see Gelfand and Shilov 1962). This method is essentially Hadamard's method for obtaining the finite part of an integral.

Let us recall that, as is well known (Titchmarch 1951), the usual Riemann zeta function, when considered as a finite part on a Hadamard integral, satisfies the usual functional equation relating $\zeta(s)$ and $\zeta(1-s)$ (for more details see the discussion in appendix A of our previously quoted paper).

In this paper, our discussion will be centred around two examples: the Casimir effect in the one-dimensional box and in the three-dimensional rectangular system. The generalisation for other cases will not present any difficulty.

2. The one-dimensional box

Let us consider a massless scalar field in a one-dimensional box of length L, with the boundary condition that the field is zero at the ends; the eigenfrequencies are given by:

$$\omega_n = \frac{\pi c}{L} n \qquad n = 1, 2, \ldots \tag{2.1}$$

where c is the wave velocity. The zero-point energy is:

$$E_0 = \frac{\hbar}{2} \sum_{n=1}^{\infty} \omega_n = \frac{1}{2} \frac{\pi \hbar c}{L} \sum_{n=1}^{\infty} n \tag{2.2}$$

which is infinite. By introducing the exponential cut-off:

$$E_0 = \frac{\hbar}{2} \lim_{\alpha \to 0^+} \sum_{n=1}^{\infty} \omega_n \, \mathrm{e}^{-\alpha \omega_n} = -\frac{\hbar}{2} \lim_{\alpha \to 0^+} \frac{\partial}{\partial \alpha} \sum_{n=1}^{\infty} \mathrm{e}^{-\alpha \omega_n} \tag{2.3}$$

or

$$E_0 = -\frac{\hbar}{2} \lim_{\alpha \to 0^+} \frac{\partial}{\partial \alpha} \sum_{n=1}^{\infty} \mathrm{e}^{-\alpha (\pi c/L) n} = \lim_{\alpha \to 0^+} \left(-\frac{\hbar L}{2\pi c} \frac{B_0}{\alpha^2} - \frac{\hbar \pi c}{4L} B_2 + \mathrm{O}(\alpha^2) \right) \tag{2.4}$$

where B_m are the Bernoulli numbers:

$$B_0 = 1 \qquad B_1 = -\tfrac{1}{2} \qquad B_2 = \tfrac{1}{6} \ldots .$$

According to our prescription the finite part is obtained by excluding in equation (2.4) all the pole terms for $\alpha \to 0^+$; then

$$E_0^R = -\frac{\pi \hbar c}{4L} B_2. \tag{2.5}$$

Now we have the identity:

$$\sum_n \omega_n^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \sum_n \mathrm{e}^{-\omega_n x} \, \mathrm{d}x \tag{2.6}$$

where n is a general index. In the present case, for ω_n given by equation (2.1), equation (2.6) can be rewritten as:

$$\sum_{n=1}^{\infty} \omega_n^{-s} = \frac{1}{\Gamma(s)} \left(\frac{L}{\pi c} \right)^s \int_0^{\infty} x^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-nx}$$

or

$$\sum_{n=1}^{\infty} \omega_n^{-s} = \left(\frac{L}{\pi c} \right)^s \zeta(s) \tag{2.7}$$

where $\zeta(s)$ is the usual Riemann zeta function defined by:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-nx} \, \mathrm{d}x = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} Z_1(x) \, \mathrm{d}x \tag{2.8}$$

for $\operatorname{Re} s > 1$. Let us note that we have:

$$\int_0^{\infty} x^{s-1} Z_1(x) \, \mathrm{d}x = \int_0^{1} x^{s-1} Z_1(x) \, \mathrm{d}x + \int_1^{\infty} x^{s-1} Z_1(x) \, \mathrm{d}x. \tag{2.9}$$

The second term on the RHS of equation (2.9) can be continued analytically for $\operatorname{Re} s \leq 1$.

Now the first term on the RHS diverges for $\operatorname{Re} s \leq 1$ because of the bad behaviour of the integrand near the origin. Recalling that:

$$Z_1(x) = \frac{1}{\mathrm{e}^x - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} x^{m-1} \tag{2.10}$$

where B_m are the Bernoulli numbers, Hadamard's finite part is obtained by the following procedure. For Re $s > 1$, we have:

$$\frac{1}{\Gamma(s)} \int_0^1 x^{s-1} Z_1(x)\,\mathrm{d}x = \frac{1}{\Gamma(s)} \left[\int_0^1 x^{s-1} \left(Z_1(x) - \frac{B_0}{x} \right) \mathrm{d}x + \frac{B_0}{s-1} \right]. \tag{2.11}$$

The right-hand side of equation (2.11) presents a pole for $s = 1$, and can be continued analytically up to Re $s > 0$. This will be the analytic continuation of the left-hand side of equation (2.11) for $0 < \mathrm{Re}\, s < 1$. Similarly, in order to continue up to Re $s < -1$, we use the expression (which is the analytic continuation in the region $-2 < \mathrm{Re}\, s < -1$)

$$\frac{1}{\Gamma(s)} \left[\int_0^1 x^{s-1} \left(Z_1(x) - \frac{B_0}{x} - B_1 - \frac{B_2}{2!} x \right) \mathrm{d}x + \frac{B_0}{s-1} + \frac{B_1}{s} + \frac{B_2}{2!(s+1)} \right]. \tag{2.12}$$

As $\Gamma(s)$ has a pole for $s = -1$, the second term of equation (2.9) will not give any contribution, while the analytic continuation of the corresponding first term, which is given by equation (2.12), will receive only a contribution from $B_2/2!(s+1)\Gamma(s)$ which for $s = -1$ gives $\zeta(-1) = -B_2/2$ and reproduces equation (2.5) through the use of equation (2.7).

3. The three-dimensional rectangular box

Let us now consider a scalar field in a three-dimensional box of volume $V = L_1 L_2 L_3$, with the boundary condition that the field vanishes at the walls.

The eigenfrequencies are

$$\omega_{n_1,n_2,n_3} = \pi \left[\left(\frac{n_1}{L_1} \right)^2 + \left(\frac{n_2}{L_2} \right)^2 + \left(\frac{n_3}{L_3} \right)^2 \right]^{1/2} \tag{3.1}$$

with $c = 1$.

By taking the exponential regularisation, i.e. writing

$$\sum_{n_1,n_2,n_3=1}^{\infty} \left[\left(\frac{n_1}{L_1} \right)^2 + \left(\frac{n_2}{L_2} \right)^2 + \left(\frac{n_3}{L_3} \right)^2 \right]^{1/2}$$

$$= -\lim_{\alpha \to 0^+} \frac{\partial}{\partial \alpha} \sum_{n_1,n_2,n_3=1}^{\infty} \exp\left\{ -\alpha \left[\left(\frac{n_1}{L_1} \right)^2 + \left(\frac{n_2}{L_2} \right)^2 + \left(\frac{n_3}{L_3} \right)^2 \right]^{1/2} \right\} \tag{3.2}$$

and using the Poisson formula, we can write (see Lukosz (1971); also Ruggiero *et al* (1977)):

$$\begin{aligned} Z_3(\alpha) = {} & \sum_{n_1,n_2,n_3=1}^{\infty} \exp\left\{ -\alpha \left[\left(\frac{n_1}{L_1} \right)^2 + \left(\frac{n_2}{L_2} \right)^2 + \left(\frac{n_3}{L_3} \right)^2 \right]^{1/2} \right\} \\ = {} & \frac{\pi V}{\alpha^3} - \frac{\pi}{4\alpha^2} (L_1 L_2 + L_1 L_3 + L_2 L_3) + \frac{1}{4} \frac{(L_1 + L_2 + L_3)}{\alpha} - \frac{1}{4} \\ & - \alpha \left\{ \pi V \sum_{m_1,m_2,m_3=-\infty}^{\infty}{}' \frac{1}{16\pi^4 l^4} - \frac{\pi}{4} L_1 L_2 \sum_{m_1,m_2=-\infty}^{\infty}{}' \frac{1}{4^{3/2} \pi^3 l_1^3} \right. \\ & - \frac{\pi}{4} L_1 L_3 \sum_{m_1,m_3=-\infty}^{\infty}{}' \frac{1}{4^{3/2} \pi^3 l_2^3} - \frac{\pi}{4} L_2 L_3 \sum \sum_{m_2,m_3=-\infty}^{\infty}{}' \frac{1}{4^{3/2} \pi^3 l_3^3} \\ & \left. + \frac{1}{4} \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \right) \right\} + \mathrm{O}(\alpha^2) = \frac{A_0}{\alpha^3} + \frac{A_1}{\alpha^2} + \frac{A_2}{\alpha} + A_3 + \alpha A_4 + \mathrm{O}(\alpha^2) \end{aligned} \tag{3.3}$$

with

$$V = L_1 L_2 L_3 \qquad l^2 = (m_1 L_1)^2 + (m_2 L_2)^2 + (m_3 L_3)^2$$
$$l_1^2 = (m_1 L_1)^2 + (m_2 L_2)^2 \qquad l_2^2 = (m_1 L_1)^2 + (m_3 L_3)^2$$
$$l_3^2 = (m_2 L_2)^2 + (m_3 L_3)^2.$$

$\Sigma'^{\infty}_{m_1, m_2, \ldots, m_n = 0}$ means that in the sum $m_1 = m_2 = \ldots = m_n = 0$ is excluded.

By taking $-\partial/\partial\alpha$ of equation (3.3) and making $\alpha \to 0^+$, after the neglect of all pole terms, we obtain the regularised expression:

$$\left\{ \sum_{n_1, n_2, n_3 = 1}^{\infty} \left[\left(\frac{n_1}{L_1}\right)^2 + \left(\frac{n_2}{L_2}\right)^2 + \left(\frac{n_3}{L_3}\right)^2 \right]^{1/2} \right\}^R = A_4. \tag{3.4}$$

Consider now the generalised zeta function:

$$\sum_{n_1, n_2, n_3 = 1}^{\infty} \pi^{-s} \left[\left(\frac{n_1}{L_1}\right)^2 + \left(\frac{n_2}{L_2}\right)^2 + \left(\frac{n_3}{L_3}\right)^2 \right]^{-s/2} = \sum_{n_1, n_2, n_3 = 1}^{\infty} \omega_{n_1, n_2, n_3}^{-s}. \tag{3.5}$$

Using equation (2.6), we have:

$$\pi^s \sum_{n_1, n_2, n_3 = 1}^{\infty} \omega_{n_1, n_2, n_3}^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} Z_3(x)\, \mathrm{d}x \tag{3.6}$$

with

$$Z_3(x) = \sum_{n_1, n_2, n_3 = 1}^{\infty} \exp\left\{ -\left[\left(\frac{n_1}{L_1}\right)^2 + \left(\frac{n_2}{L_2}\right)^2 + \left(\frac{n_3}{L_3}\right)^2 \right]^{1/2} x \right\}. \tag{3.7}$$

As before, we rewrite equation (3.6) as:

$$\pi^{-s} \sum_{n_1, n_2, n_3 = 1}^{\infty} \omega_{n_1, n_2, n_3}^{-s} = \frac{1}{\Gamma(s)} \int_0^1 x^{s-1} Z_3(x)\, \mathrm{d}x + \frac{1}{\Gamma(s)} \int_1^{\infty} x^{s-1} Z_3(x)\, \mathrm{d}x. \tag{3.8}$$

The second term on the RHS of this last equation is analytic in s, while the first is only convergent for Re $s > 3$, since $Z_3(x)$, by equation (3.3), has a pole of third order in $x = 0$. For Re $s > 3$ we can write:

$$\frac{1}{\Gamma(s)} \int_0^1 x^{s-1} Z_3(x)\, \mathrm{d}x = \frac{1}{\Gamma(s)} \left[\int_0^1 x^{s-1} \left(Z_3(x) - \frac{A_0}{x^3} \right) \mathrm{d}x + \frac{A_0}{s-3} \right] \tag{3.9}$$

where A_0 is defined in equation (3.3).

The right-hand side of equation (3.9) can be continued analytically up to Re $s > 2$, exhibiting a pole at $s = 3$. In this region it defines the finite part of the left-hand side of equation (3.9).

In a similar way, the analytic continuation of the left-hand side of equation (3.9) in the region $-2 < \mathrm{Re}\, s < -1$ is given by (with the A_m defined in equation (3.3)):

$$\frac{1}{\Gamma(s)} \left[\int_0^1 x^{s-1} \left(Z_3(x) - \frac{A_0}{x^3} - \frac{A_1}{x^2} - \frac{A_2}{x} - A_3 - A_4 x \right) \mathrm{d}x \right.$$
$$\left. + \frac{A_0}{s-3} + \frac{A_1}{s-2} + \frac{A_2}{s-1} + \frac{A_3}{s} + \frac{A_4}{s+1} \right]. \tag{3.10}$$

For more details see the book of Gelfand and Shilov (1962).

Expression (3.10) is well defined at $s = -1$, since the pole at this point is cancelled by the corresponding one of $\Gamma(s)$. As $[\Gamma(-1)]^{-1} = 0$, the analytic continuation of equation

(3.6) at $s=-1$ is exactly A_4. In our previous paper, A_4 was expressed in terms of the Epstein zeta functions (Epstein 1902, appendix B of Ruggiero *et al* 1977).

Let us remark that the analytic continuation using the method of Hadamard gives finite parts from divergent integrals. This is most easily seen by looking at equation (2.11). There we have added to the right-hand side the term

$$-\frac{1}{\Gamma(s)}\frac{B_0}{s-1}\epsilon^{s-1}$$

with $\epsilon \to 0^+$ which comes from the lower end of the integral. For Re $s>1$, this is evidently zero, but for Re $s<1$ it is infinite.

Therefore it is not surprising that, in the examples discussed by us, the zeta regularisation gives automatically finite results for physical values of s (although in some other examples we can obtain poles for physical values of this parameter, as happens in the case of second-order electromagnetic self-energy of a non-relativistic electron between two conducting parallel plates).

Finally we would like to mention two recent preprints (Ford 1979, Toms 1979) which use the zeta function techniques in problems very similar to those discussed in our previous paper.

They criticise the application of exponential cut-offs $\Sigma_n \mathrm{e}^{-\alpha\omega_n}$ (used by Casimir (1948) and Fierz (1960)) for the study of the Casimir effect in curved space, because of the ambiguity in the subtraction of infinite quantities. For this reason they prefer the use of zeta function regularisation which would be free of these ambiguities (see also Hawking 1977).

From our point of view, if we interpret the parameter α as an analytic regulator (Re $\alpha>0$) in the 'partition function' $\Sigma_n \mathrm{e}^{-\alpha\omega_n}$, then in order to obtain the finite part of it we subtract all the pole terms in $\alpha=0$. We have shown here in some relevant examples that this procedure is mathematically equivalent to the analytic continuation of the corresponding zeta function.

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